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P_S-Connected Spaces

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Abstract

The aim of this work is to introduce a new generalization of a pre-connected space, which is called P_S-connected space and study relationship with other types of connected spaces and gives some results on it. Further results concerning preservation of this connectedness like properties under surjections are obtained.

1. Introduction

In 2007 Khalaf A. B. and Asaad B. A. introduced a new class of preopen sets called the class of P_S-open sets ***[13]***, while preopen sets was introduced by Mashhour et.al. at 1982 ***[5]***. In 1978 Cameron D. E. defined regular semi-open sets ***[12]***. V. Popa in 1987 defined the concept of p connected or preconnected spaces ***[2]***.

Throughout this paper X and Y will always denote topological spaces on which no separation axioms are assumed unless explicitly stated. If A is a subset of X, then the closure of A and interior of A are denoted by cl(A) and int(A) respectively. The P_S-closure of A is the intersection of all P_S-closed sets containing A***[13]*** while pcl(A) denote the preclosure of A***[8]***. The symbols sint(A) and scl(A) denote the semi-interior and semi-closure of A, respectively ***[11]***.

2. Preliminaries

The aim of this section is to gives some definitions, theorems and results that we used in the next section.

Definitions 2.1: A subset A of a space X is said to be pre-regular [4] (resp. regular semi-open [12]) if A is both preopen and preclosed set (resp. $A = \mathrm{sint}(\mathrm{scl}(A))$).

Definitions 2.2: A subset A of a space X is said to be preopen [5] (resp. regular open, regular closed [1], β-open [14]) if $A \subseteq \text{intcl}(A)$ (resp. $A = \text{intcl}(A)$, $A = \text{clint}(A)$ and $A \subseteq \text{clintcl}(A)$). The family of all preopen, regular open, regular closed and β-open denoted by $PO(X), RO(X), RC(X)$ and $\beta O(X)$.

Definitions 2.3: A space X is said to be connected [2] (resp. preconnected, β-connected [7]) if X cannot be written as a union of two non-empty disjoint open (resp. preopen, β-open) sets.

Definitions 2.4: A space X is said to be PS-space [7] (resp. locally indiscrete [9]) if every preopen (resp. open) set in X is semi-open (resp. closed).

Definitions 2.5: A function $f: X \to Y$ is said to be continuous [2] (resp. P_s-continuous [13], totally (= Perfectly) continuous [10] and completely continuous [3]) if the inverse image every open set in Y is open (resp. P_s-open, clopen and regular open) set in X.

Lemma 2.6 [7]: Every β-connected space is preconnected.

Theorem 2.7 [7]: A space X is preconnected if and only if X is connected and PS-space.

Definition 2.8 [6]: A space X is said to be semi-T1 if to each pair of distinct points x and y in X, there exist a pair of semi-open sets, one containing x, but not y and the other containing y but not x.

Theorem 2.9 [2]: A space X is disconnected if and only if X is the union of two non-empty disjoint open sets.

The following definitions, theorems, propositions and lemmas are found in **[13]**:

Definition 2.10: A subset A of a space X is said to be P_s-open if it is preopen and for each $x \in A$, there exists a semi-closed set F such that $x \in F \subseteq A$. The family of all P_s-open sets denoted by $P_sO(X)$.

Proposition 2.11: Let A and Y be subsets of a space X such that $A \subseteq Y$. If Y is open or (pre-regular or regular semi-open) subspace of X, then $P_s\text{cl}_Y(A) \subseteq P_s\text{cl}(A)$, where $P_s\text{cl}_Y(A)$ is the P_s-closure of A relative to the subspace Y.

Proposition 2.12: Let Y be a regular open subspace of a space X and $A \in P_sO(Y)$. Then $P_s\text{cl}(A) \subseteq P_s\text{cl}_Y(A)$, where $P_sO(Y)$ denote the set of all P_s-open sets in Y.

Lemma 2.13: Let A and B be subsets of a space X.If $A \subseteq B$, then $P_s\text{cl}(A) \subseteq P_s\text{cl}(B)$.A is P_s-closed set in X if and only if $P_s\text{cl}(A) = A$.

Proposition 2.14: If a space X is semi-T1, then preopen sets and P_s-open sets in X are identical.

Lemma 2.15: Every clopen and regular open set in a space X is P_s-open.

Proposition 2.16: If a space X is locally indiscrete, then P_s-open sets and open sets in X are identical.

Proposition 2.17: A function $f: X \to Y$ is P_s-continuous if and only if the inverse image of every closed set in Y is P_s-closed in X.

Theorem 2.18: If a function $f: X \rightarrow Y$ is open and continuous function, then $f^{-1}(A)$ is P_s-open sets in X, for each P_s-open set A in Y.

3. P_s-Connected Spaces

Definition 3.1: Let X be a space. Two non-empty subsets A and B of X are said to be P_s-separated sets in X or ($\tau - P_s$-separated) if $P_s cl(A) \cap B = \phi$ and $A \cap P_s cl(B) = \phi$.

Example 3.2: Consider $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{c\}, \{a, d\}, \{a, c, d\}\}$. Then $P_s O(X) = \{\phi, X, \{c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$, here $\{a\}$ and $\{b\}$ are P_s-separated sets in X since $P_s cl(\{a\}) \cap \{b\} = \{a\} \cap \{b\} = \phi$ and $\{a\} \cap P_s cl(\{b\}) = \{a\} \cap \{b\} = \phi$.

Lemma 3.3: Every preconnected space if P_s-connected.

Proof:

Let X be preconnected space. Then by Definition 2.3, X can not be expressed as a union of two non-empty disjoint preopen sets. To show X is P_s-connected if possible suppose that X is P_s-disconnected, then there exist two non-empty disjoint P_s-open sets A and B such that $X = A \cup B$, and since every P_s-open set is preopen set, so A and B are non-empty disjoint preopen sets such that $X = A \cup B$ implies that X is not preconnected space, which is a contradiction. Thus, X is P_s-connected space.

The converse of the Lemma 3.3 is not true in general as shown in the following example:

From Example 3.2, $PO(X) = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\},$

$\{b, c, d\}, \{d\}, \{a, c, d\}\}$ and $P_s O(X) = \{\phi, X, \{c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$, then X is P_s-connected space, but not preconnected since there exist two non-empty disjoint preopen sets $\{a, b, c\}$ and $\{d\}$ such that $X = \{a, b, c\} \cup \{d\}$.

Corollary 3.4: Every β-connected space is P_s-connected.

Proof: Follows from Lemma 2.6 and Lemma 3.3.

Corollary 3.5: Every connected and PS-space is P_s-connected.

Proof: Follows form Theorem 2.7.

Lemma 3.6: Every disconnected space is P_s-disconnected.

Proof: Follows from the fact that every clopen sets is P_s-open.

From this proposition we get that every P_s-connected space is connected. But the converse is not true as shown in the following example:

Example 3.7:[14] Let (R, τ_{cof}) be the co-finite topological space. Then R is connected but not P_s-connected since R is the union of two non-empty disjoint P_s-open sets which are Q = the set of all rational numbers and Irr = the set of all irrational numbers.

Theorem 3.8: Let X be a space, Y an open or (pre-regular or regular semi-open) subspace of X and $A, B \subseteq Y$. If A and B are P_s-separated sets in X, then A and B are $\tau_Y - P_s$-separated sets in Y.

Proof:

Let A and B be $\tau - P_s$-separated sets. Then $P_s cl(A) \cap B = \phi$ and $A \cap P_s cl(B) = \phi$, but since Y is open or (pre-regular or regular semi-open) subspace of X so by Proposition 2.11, $P_s cl_Y(A) \subseteq P_s cl(A)$ and $P_s cl_Y(B) \subseteq P_s cl(B)$ implies that $P_s cl_Y(A) \cap B = \phi$ and$A \cap P_s cl_Y(B) = \phi$. Thus, A and B are $\tau_Y - P_s$-separated sets in Y.

Theorem 3.9: Let Y be a regular open subspace of a space X and A, B be two P_s-open subsets of Y. If A and B are $\tau_Y - P_s$-separated in Y, then they are $\tau - P_s$-separated in X.

Proof:

Let A and B be $\tau_Y - P_s$-separated sets in Y. Then $P_s cl_Y(A) \cap B = \phi$ and $A \cap P_s cl_Y(B) = \phi$ and since Y is regular open set in Y and A, B are P_s-open sets in Y, so by Proposition 2.12, $P_s cl(A) \subseteq P_s cl_Y(A)$ and $P_s cl(B) \subseteq P_s cl_Y(B)$. This implies that $P_s cl(A) \cap B = \phi$ and $A \cap P_s cl(B) = \phi$. Therefore, A and B are $\tau - P_s$-separated in X.

Proposition 3.10: If A and B are two P_s-open subsets of a space X. Then they are P_s-separated if and only if they are disjoint.

Proof:

Necessity: Let A and B be two disjoint P_s-open subsets in a space X. Since $A \cap B = \phi$, then $A \subseteq X\backslash B$ and $B \subseteq X\backslash A$ implies that by Lemma 2.13, $P_s cl(A) \subseteq P_s cl(X\backslash B)$ and $P_s cl(B) \subseteq P_s cl(X\backslash A)$. But $X\backslash A$ and $X\backslash B$ are P_s-closed sets in X, and then $P_s cl(X\backslash A) = X\backslash A$ and $P_s cl(X\backslash B) = X\backslash B$. This implies that $P_s cl(A) \cap B = \phi$ and $P_s cl(B) \cap A = \phi$. Hence A and B are P_s-separated sets.

Sufficiency: Obvious.

Definition 3.11: A space X is said to be P_s-connected if it cannot be expressed as a union of two non-empty proper P_s-separated sets in X. Otherwise X is called P_s-disconnected.

Proposition 3.12: Let X be a space. Then X is P_s-connected if and only if it cannot be written as a union of two non-empty disjoint P_s-open sets.

Proof:

Let X be P_s-connected space and if possible suppose that there exists two disjoint non-empty P_s-open sets A and B such that $X = A \cup B$. Then by Proposition 3.10, A and B are P_s-separated sets this implies that X is not P_s-connected space, which is a contradiction. Thus, X can not be expressed as a union of two non-empty disjoint P_s-open sets.

Conversely, let the hypothesis be satisfied and X be P_s-disconnected space. Then, there exist two non-empty P_s-separated sets A and B such that $X = A \cup B$ implies that $P_s cl(A) \cap B = \phi$ and then $A \cap B = \phi$. But $P_s cl(A) \subseteq X\backslash B = A$ and since $A \subseteq P_s cl(A)$ this implies that $A = P_s cl(A)$, therefore by Lemma 2.13, A is a P_s-closed set in X and by the same way B is P_s-closed set in X and since $A \cap B = \phi$ and $X = A \cup B$ implies that A and B are P_s-open sets in X. Thus, X is expressed as a union of non-empty disjoint P_s-open sets which is a contradiction, implies that X is P_s-connected space.

Corollary 3.13: A space X is P_s-connected if and only if X cannot be expressed as a union of two non-empty disjoint P_s-closed sets.

Proposition 3.14: Let X be a space. Then X is P_s-connected space if and only if there is no non-empty proper subset of X which is both P_s-open and P_s-closed sets.

Proof:

Let X be a P_s-connected space and suppose that there exists a non-empty proper subset A of X which is both P_s-open and P_s-closed. Then $B = X\backslash A$ is a non-empty proper subset of X which is both P_s-open and P_s-closed. But $P_s cl(A) \cap B = A \cap B = \phi$ and $A \cap P_s cl(B) = A \cap B = \phi$ implies that A and B are P_s-separated sets and $X = A \cup B$, then X is not P_s-connected which is a contradiction. Thus there no non-empty proper subset of X which is both P_s-open and P_s-closed set

Conversely, let the hypothesis be hold. To show X is P_s-connected space, if possible suppose that X is P_s-disconnected, then by Proposition 3.12, there exist two non-empty disjoint P_s-open sets A and B such that $X = A \cup B$. Now since $A \cap B = \phi$ implies that $A = X\backslash B$ is P_s-closed set; therefore A is a non-empty proper subset of X which is both P_s-open and P_s-closed set, which is a contradiction of the hypothesis. Thus, X must be a P_s-connected space.

Corollary 3.15:A space X is P_s-connected if and only if the only non-empty subset of X, which is both P_s-open and P_s-closed is X itself.

Proposition 3.16: If a space X is P_s-connected semi-T1, then it is preconnected.

Proof: The proof follows directly since by Proposition 2.14, preopen sets and Ps-open sets are identical.

The hereditary property about P_s-connected is not hold in general as an example below:

Example 3.17: Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Then $P_sO(X) = \{\phi, X\}$ implies that by Corollary 3.15, X is P_s-connected. Now, let $Y = \{a, c\}$, then $\tau_Y = \{\phi, Y, \{a\}, \{c\}\}$ and $P_sO(Y) = \tau_Y$. Here Y is not P_s-connected since it can be written as a union of two non-empty disjoint P_s-open sets in Y.

Theorem 3.18: If a P_s-connected set in a space X is a subset of the union of two P_s-separated sets in X, then it is contained in one of them.

Proof:

Let A be a P_s-connected set in a space X and C, D be P_s-separated sets in X such that $A \subseteq C \cup D$. To show either $A \subseteq C$ or $A \subseteq D$, if possible suppose that $A \nsubseteq C$ and $A \nsubseteq D$. Now if $A \cap C = \phi$ and $A \cap D \neq \phi$, then since $A \subseteq C \cup D$ implies that $A \subseteq D$ which is a contradiction since $A \nsubseteq D$. By the same way we get a contradiction for the case $A \cap C \neq \phi$ and $A \cap D = \phi$. If $A \cap C \neq \phi$ and $A \cap D \neq \phi$ implies that $A = (A \cap C) \cup (A \cap D)$ and since C, D are P_s-separated sets in X, so $P_s cl(C) \cap D = \phi$ and $C \cap P_s cl(D) = \phi$.

Now,$(A \cap C) \cap P_s cl(A \cap D) \subseteq (A \cap C) \cap P_s cl(D) = A \cap \left(C \cap P_s cl(D)\right) = \phi$ this implies that $(A \cap C) \cap P_s cl(A \cap D) = \phi$ and similarly $P_s cl(A \cap C) \cap (A \cap D) = \phi$ and then $A \cap C$ and $A \cap D$ are P_s-separated sets in X such that $A = (A \cap C) \cup (A \cap D)$ this implies that A can be expressed as a union of two non-empty disjoint P_s-separated sets in X, then X is not P_s-connected which is a contradiction. Hence in each case we get a contradiction. Thus, either $A \subseteq C$ or $A \subseteq D$.

Proposition 3.19:A space X is P_s-connected if any two elements x and y in X are contained in a P_s-connected subspace of X.

Proof:

Let X be not P_s-connected space. Then by Proposition 3.12, there exist two non-empty disjoint P_s-open sets A and B such that $X = A \cup B$. Now, since A and B are non-empty, so there exists $a \in A$ and $b \in B$ implies that by hypothesis a and b are contained in some P_s-connected subspace say Y of X, but $X = A \cup B$, then $Y \subseteq A \cup B$ and then by Theorem 3.18, either $Y \subseteq A$ of $Y \subseteq B$ implies that a, b are either in A or are both in B, which is a contradiction. Hence X is a P_s-connected space.

Proposition 3.20: If A is a P_s-connected set in a space X such that $A \subseteq B \subseteq P_s cl(A)$, then B is also a P_s-connected in X.

Proof:

Let B be P_s-disconnected set in X. Then there exist two P_s-separated sets U and V such that $B = U \cup V$, but $A \subseteq B$ implies that $A \subseteq U \cup V$. Since A is P_s-connected set in X, then by Theorem 3.18, either $A \subseteq U$ or $A \subseteq V$. Now if $A \subseteq U$, then by Lemma 2.13, $P_s cl(A) \subseteq P_s cl(U)$ and since U and V are P_s-separated sets so $P_s cl(A) \cap V = \phi$, but $B \subseteq P_s cl(A)$ implies that $B \cap V = V = \phi$ which is a contradiction. Also if $A \subseteq V$ by the same way we get a contradiction. Thus B is must be P_s-connected set in X.

Corollary 3.21: The P_s-closure of any P_s-connected set in a space X is also P_s-connected set in X.

Proposition 3.22: If for every non-empty subset A of a space X, $P_s cl(A) = X$, then X is P_s-connected.

Proof:

Let the hypothesis be holds and suppose that X is P_s-disconnected space. Then by Proposition 3.12, there exists two non-empty disjoint P_s-open sets A and B such that $X = A \cup B$. Then, $A = X \backslash B$ and $B = X \backslash A$ and they are also non-empty P_s-closed sets in X. Then, $P_s cl(A) = A \neq X$ and $P_s cl(B) = B \neq X$ which is a contradiction to the hypothesis. Thus, X must be P_s-connected space.

Proposition 3.23: A locally indiscrete connected space is P_s-connected.

Proof: Follow from Theorem 3.12 and Proposition 2.16.

Theorem 3.24: The union of a collection of P_s-connected sets in a space X, where their intersection is non-empty, is also P_s-connected set in X.

Proof:

Let $\{V_\alpha : \alpha \in \Delta\}$ be a collection of P_s-connected sets in X such that $\bigcap_{\alpha \in \Delta} V_\alpha \neq \phi$ and suppose that $\bigcup_{\alpha \in \Delta} V_\alpha$ be not P_s-connected set. Then $\bigcup_{\alpha \in \Delta} V_\alpha$ can be expressed as a union of two non-empty disjoint P_s-separated sets A and B, implies that $\bigcup_{\alpha \in \Delta} V_\alpha = A \cup B$. Now since for each $\alpha \in \Delta$, $V_\alpha \subseteq \bigcup_{\alpha \in \Delta} V_\alpha$ implies that $V_\alpha \subseteq A \cup B$, but since for each $\alpha \in \Delta$, V_α is P_s-connected set so by Theorem 3.18, either $V_\alpha \subseteq A$ or $V_\alpha \subseteq B$ for each $\alpha \in \Delta$. If $V_\alpha \subseteq A$ for each $\alpha \in \Delta$, then $\bigcup_{\alpha \in \Delta} V_\alpha \subseteq A$ which is a contradiction to the assumption that A and B are P_s-separated sets in X, and by the same way if $V_\alpha \subseteq B$ for each $\alpha \in \Delta$ we get a contradiction. Thus $\bigcup_{\alpha \in \Delta} V_\alpha$ must be P_s-connected set in X.

Proposition 3.25: A space X is P_s-connected if and only if each P_s-continuous function from X into a discrete two point space $\{a, b\}$ is constant.

Proof:

Let X be a P_s-connected space and $f: X \to \{a, b\}$ be a P_s-continuous function, where $\{a, b\}$ is a discrete space. Now since f is P_s-continuous and $\{a, b\}$ is a discrete space so by Definition 2.5 and Proposition 2.17, for each $y \in f(X) \subseteq \{a, b\}$, $f^{-1}(\{y\})$ is non-empty P_s-clopen in X. But since X is P_s-connected space, so by Corollary 3.15 $f^{-1}(\{y\}) = X$ this implies that $f(x) = y$ for all $x \in X$, therefore; f is constant function.

Conversely, let the hypothesis be satisfied and suppose that X be P_s-disconnected space. Then by Proposition 3.14, there exists a non-empty proper subset A of X which is P_s-clopen, this implies that $X\backslash A$ is also a non-empty proper P_s-clopen subset of X. Now define a function $f: X \to \{a, b\}$ by $f(x) = a$ if $x \in A$ and $f(x) = b$ if $x \in X\backslash A$. Since $\{a, b\}$ is discrete space and $A \cap (X\backslash A) = \phi$, then $f^{-1}(\{a\}) = A$ and $f^{-1}(\{b\}) = X\backslash A$ and clearly $f^{-1}(\phi) = \phi$, $f^{-1}(\{a, b\}) = X$ this implies by Definition 2.5, f is P_s-continuous. But f is not constant function, which is a contradiction of the hypothesis. Hence, X must be P_s-connected.

Proposition 3.26: A P_s-continuous image of a P_s-connected space is connected.

Proof:

Let $f: X \to Y$ be a P_s-continuous and X be P_s-connected. To show f(X) is connected in Y, if possible suppose that Y is disconnected space. Then by Theorem 2.9 f(X) is the union of two non-empty disjoint open sets A and B in Y, since f is P_s-continuous function so by Definition 2.5 $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty disjoint P_s-open sets in X. But $f(X) = A \cup B$ implies that $X = f^{-1}(A) \cup f^{-1}(B)$ and then X is the union of two non-empty disjoint P_s-open sets in X implies that X is P_s-disconnected which is a contradiction. Thus f(X) must be connected set in Y.

Corollary 3.27: Let f be a surjective P_s-continuous from a P_s-connected space X to a space Y. Then Y is connected space.

Proof: Follow from Proposition 3.26.

Theorem 3.28: If $f: X \to Y$ is a surjective open continuous function, where X is P_s-connected space, then Y is also P_s-connected.

Proof:

Let Y be P_s-disconnected space. Then by Proposition 3.12, there exist two non-empty disjoint P_s-open sets A and B such that $Y = A \cup B$, since f is continuous and open function, so by Theorem 2.18, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty disjoint P_s-open sets in X such that $X = f^{-1}(A) \cup f^{-1}(B)$ this implies that by Proposition 3.12 X is P_s-disconnected space which is a contradiction. Thus Y must be P_s-connected space.

Proposition 3.29: If f is a surjective totally (perfectly) continuous function from a P_s-connected space X to a space Y, then Y is connected.

Proof:

If possible suppose that Y is disconnected, then by Theorem 2.9, there exist two non-empty disjoint open sets A and B in Y such that $Y = A \cup B$. Now since f is totally (perfectly) continuous function, so by Definition 2.5, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty disjoint clopen sets in X, and by Lemma 2.15, $f^{-1}(A)$ and $f^{-1}(B)$ are P_s-open sets in X, but $f(X) = Y = A \cup B$ implies that $X = f^{-1}(A) \cup f^{-1}(B)$ and then X is P_s-disconnected which is a contradiction. Thus, Y must be connected.

Proposition 3.30: If f is a surjective completely continuous function from a P_s-connected space X to a space Y, then Y is connected.

Proof: Follow from Theorem 2.9, Proposition 3.12, Definition 2.5 and Lemma 2.15.

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